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The isomorphism problem for commutative monoid rings[1]

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Abstract

By substantial changes and corrections in Demushkin's old paper the essentially final positive answer to the isomorphism problem for monoid rings of submonoids of $\mathbb{Z}^r$ is obtained. This means that the underlying monoid is shown to be determined (up to isomorphism) by the corresponding monoid ring. Thereafter the positive answer to the analogous question for the 'dual' objects – descrete Hodge algebras – is derived. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The isomorphism problem for monoid rings asks whether two monoids are isomorphic if they have isomorphic monoid rings (with coefficients in some ring). Here we are concerned with the case of commutative monoids and commutative rings.

This paper contains an essentially final positive solution of the isomorphism problem for finitely generated, commutative, cancellative and torsion free monoids (Theorem 2.1). But some remarks are in order.

The problem is mentioned in the very last section of Gilmer's book [11]. For a decade there was no work on this problem, and it is only recently that papers related to the isomorphism problem have appeared, such as [12, 13] (related mostly to the non-commutative case) and [18] (where we give a positive answer in the special case of finitely generated submonoids of $\mathbb{Z}^2$). All the authors mentioned above have remarked that there is no previous study of the isomorphism problem dated before [11].

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It turns out, however, that as early as 1982 Demushkin claimed the positive answer to the isomorphism problem for all *finitely generated normal* monoids *without non-trivial units* (for definitions see Section 2).

The proof as presented in [9] contains several 'spurious arguments' (Math. Rev. 84f : 14036). However, it is our goal in the present paper to show that all these spurious arguments can be changed by the correct ones, restoring the proof (applicable to the general case of *not necessarily normal* monoids).

I have reorganized the material in such a way that *general* (not necessarily normal) monoids could be involved. The proof is carried out in 'pure commutative algebraic' terms and only the basic background is required for reading it (except, maybe, Borel's theorem on maximal tori in linear algebraic groups). For the origin of our "toric" terminology the reader is referred to [7, 10, 20].

In Section 3 an application to the 'isomorphism problem for discrete Hodge algebras' is presented (that for two isomorphic discrete Hodge algebras the corresponding defining monomial ideals are shown to be the same modulo a suitable bijective correspondence of the variables of reference).

Below $\mathbb{N}=\{1,2,\dots\}$, $\mathbb{Z}=\{0,\pm 1,\pm 2,\dots\}$, and # refers to the number of elements in the corresponding set.

I would like to thank N.V. Trung for bringing to my attention [9] and B. Totaro for pointing out the rôle of Proposition 2.16 in circumventing one of the essential gaps in [9] (not mentioned in Math. Rev. 84f : 14036 however). Other essential changes include Step 3 and Step 7 in [9] (Steps 1 and 3 in this paper respectively), and our use of $GL(\mu/\mu^d)$ instead of $GL(\mu/\mu^2)$. (See also Comments 2.17.)

This paper was written during my stay at the University of Chicago, Fall 1995. I would like to take this opportunity thank Richard G. Swan for my invitation, and for his strong support during the years. (He also noticed one more hidden gap in Demushkin's arguments that led us to the aforementioned use of the groups $GL(\mu/\mu^d)$ with d large.)

2. The isomorphism problem

All the considered monoids are assumed to be commutative, cancellative and torsion free, that is, the natural monoid homomorphisms

$$M \to K(M) \to \mathbb{Q} \otimes K(M)$$

are assumed to be injective, where $K(M)$ denotes the group of quotients of a monoid M and $\mathbb{Q}$ is the additive group of rational numbers.

For a monoid M its biggest subgroup, the group of units, is denoted by $U(M)$. If $U(M)$ is trivial, then for any ring R the monoid ring $R[M]$ carries naturally an augmented R-algebra structure. Namely, we consider the augmentation $R[M] \to R$ under which all nontrivial elements of M map to $0 \in R$.

Recall that a monoid M is called normal if (writing additively) $c \in \mathbb{N}$, $x \in K(M)$, $cx \in M$ imply $x \in M$. It is well known that M is normal iff for any (equivalently, some) field k the monoid domain $k[M]$ is integrally closed [11, Corollary 12.6].

Theorem 2.1. *Let M and N be two finitely generated monoids and R a ring. Assume $R[M] \approx R[N]$ as R-algebras. Then*

(a) *$M \approx N$ if $U(M)$ and $U(N)$ are trivial and $R[M] \approx R[N]$ as augmented R-algebras,*
(b) *$M \approx N$ if M is normal,*
(c) *$M \approx N$ if M is homogeneous.*

(Homogeneous monoids are defined in Step V – Substep 2 below; they include polytopal semigroups from [4].)

We need some preparatory work.

Step I. As mentioned above, for a monoid M we have the natural embeddings

$$M \to K(M) \to \mathbb{Q} \otimes K(M) \approx \oplus_r \mathbb{Q} \to \oplus_r \mathbb{R},$$

where $\mathbb{R}$ refers to the real numbers. The cardinal number r is called the rank of M and it is denoted by $rank(M)$. We shall always assume $rank(M) < \infty$. Thus our monoids can be thought of as additive submonoids of finite-dimensional rational vector spaces.

For a monoid M we let $C(M)$ denote the convex cone (in the corresponding real vector space) spanned by M and with vertex at the origin $O \in \oplus_r \mathbb{R}$.

Lemma 2.2. *Let M be a monoid with trivial $U(M)$. Then the following conditions are equivalent:*

(a) *$K(M)$ is finitely generated and $C(M)$ is a finite rational convex polyhedral cone,*
(b) *M is finitely generated.*

Proof (*sketch*). (b) $\Rightarrow$ (a) is obvious. For (a) $\Rightarrow$ (b) we decompose $C(M)$ into rational simplicial cones and use the special case of the implication (a) $\Rightarrow$ (b) when $C(M)$ is simplicial (the latter proved by rather straightforward arguments). □

The implication (a) $\Rightarrow$ (b) is known as Gordan's Lemma. We remark that the special case when M is normal is considerably easier [7, Section 8].

Let M be a finitely generated monoid with $U(M)$ trivial and let $C(M)$ be as above. We denote by $\Phi(M)$ the transversal section of the cone $C(M)$ by any suitable hyperplane in the real vector space of reference. $\Phi(M)$ is a finite convex polyhedron determined up to projective equivalence [14–16]. In this situation for any nontrivial element $x \in M$ we let $\Phi(x)$ denote the point of intersection of $\Phi(M)$ with the radial ray determined by x (clearly, everything is determined up to projective transformation). We have $\dim(\Phi(M)) = rank(M) - 1$.

For a monoid M and its element x we denote by $x^{-1}M$ the localization of M with respect to x, i.e., $x^{-1}M$ is the smallest submonoid of $K(M)$ containing M and $-x$ (writing additively). Thus,

$$x^{-1}M = \{y + cx \mid y \in M, c \in \mathbb{Z}\} \subset K(M).$$

Lemma 2.3. *Let M be a finitely generated normal monoid with $U(M)$ trivial and let $x \in M$, $x \neq 0$. Assume $\Phi(x)$ is a vertex of $\Phi(M)$. Then*

$$x^{-1}M \approx \mathbb{Z} \times M'$$

for some finitely generated normal monoid M' with $U(M')$ trivial; furthermore, $\Phi(M')$ is a $(\dim\Phi(M) - 1)$-dimensional polyhedron obtained by some transversal section of the cone with vertex $\Phi(x)$ spanned by $\Phi(M)$.

Proof. See [14, Theorem 1.8] and the proof of Proposition 2.6 in [17]. □

Let M be a finitely generated monoid with $U(M)$ trivial. Assume $W \subset \Phi(M)$ is a convex subset. Then we put $M(W) = \{x \in M \mid x \neq 0, \Phi(x) \in W\} \cup \{0\}$. $M(W)$ is a submonoid of M. In case W is a rational convex subpolyhedron of $\Phi(M)$ (i.e., the vertices of W are of the type $\Phi(x)$ for some $x \in M$), the submonoid $M(W) \subset M$ is finitely generated. This follows from Lemma 2.2.

We recall that a pyramid is a finite convex polyhedron Δ which is a convex hull of a point v and some convex polyhedron P of dimension $\dim(\Delta) - 1$ (equivalently, v does not belong to the affine hull of P in the ambient real vector space). In this situation v is called a vertex of Δ and P is called the base of Δ opposite to v.

Let M be a monoid and $x \in M$. We shall say that x *splits* M if there exists a submonoid $N \subset M$ such that the homomorphism

$$\mathbb{Z}_+ \times N \to M,$$

defined by $(c, y) \mapsto cx + y$, is an isomorphism ($\mathbb{Z}_+$ denotes the additive monoid of nonnegative integers).

If x splits M, then for any ring R the monoid ring $R[M]$ is naturally R-isomorphic to the polynomial ring $R[N][X]$ (N as above). In other words x is a variable for $R[M]$.

Observe that if x splits a monoid M, the aforementioned submonoid $N \subset M$ is uniquely determined (notwithstanding whether $U(M)$ is trivial or not). Indeed, N is precisely the set of those elements $m \in M$ such that $m - x$ is not in M.

For x and N as above, N will be called a *complementary* monoid of x.

Lemma 2.4. *Let k be a field and M be a finitely generated monoid with $U(M)$ trivial. Assume $z \in M \setminus \{0\}$ and $\Phi(M)$ is a pyramid so that $\Phi(z)$ is its vertex. Let B denote the base of $\Phi(M)$ opposite to $\Phi(z)$. Then the following assertions are equivalent:*

(a) $\sqrt{zk[M]} \subset k[M]$ *is a principal ideal,*

(b) *there is an element $x \in M \setminus \{0\}$ splitting M for which $M(B)$ is a complementary monoid and $\Phi(x) = \Phi(z)$.*

Proof. (a) $\Rightarrow$ (b). Since $z \in \sqrt{zk[M]}$, standard arguments with Newton polyhedra show that

$$\sqrt{zk[M]} = xk[M]$$

for some $x \in M \setminus \{1\}$ (here the monoid operation is written multiplicatively as we always do when the monoid of reference is considered in the corresponding monoid ring). Since $\Phi(z)$ is a vertex of $\Phi(M)$, by easy geometric arguments we see that

$$\sqrt{zk[M]} = Ker(k[M] \xrightarrow{\pi} k[M(B)]),$$

where π is the k-algebra retraction determined by $\pi(y) = y$ for $y \in M(B)$ and $\pi(y) = 0$ for $y \in M \setminus M(B)$.

Therefore, any element $y \in M \setminus M(B)$ admits a representation of the type

$$y = xy_1$$

for some $y_1 \in M$. If $y_1 \in M \setminus M(B)$ we find $y_2 \in M$ such that

$$y_1 = xy_2$$

and so on. Hence we obtain a strictly increasing sequence of principal ideals

$$(y) \subset (y_1) \subset (y_2) \subset \cdots$$

which must stop since $k[M]$ is Noetherian. This implies that M is generated (as a monoid) by x and $M(B)$. Now the vector x does not belong to the real vector space $\mathbb{R} \otimes K(M(B))$ (both sitting in the real space $\mathbb{R} \otimes K(M)$). This shows (b)

(b) $\Rightarrow$ (a). We leave this to the reader as an easy exercise. $\square$

Lemma 2.5. *Let k be a field and M be a finitely generated normal monoid with $U(M)$ trivial. Assume $\Phi(M)$ is a pyramid with vertex $\Phi(z)$ for some non-unit (in multiplicative terminology) $z \in M$. Assume further that the natural homomorphism of divisor class groups $Cl(k[M]) \to Cl(k[M]_z)$ is an isomorphism. Then M is split by some non-unit $x \in M$ for which $\Phi(x) = \Phi(z)$. Moreover, the element x is defined uniquely.*

Proof. By Lemma 2.4 everything is reduced to showing that

$$\sqrt{zk[M]} \subset k[M]$$

is a principal ideal (the uniqueness of x is clear).

As mentioned in the proof of Lemma 2.4,

$$\sqrt{zk[M]} = Ker(k[M] \xrightarrow{\pi} k[M(B)])$$

(notation as in the mentioned proof). In particular, $\sqrt{zk[M]}$ is a height 1 prime ideal of $k[M]$ and, hence, defines an element of $Cl(k[M])$. The claim above amounts to showing that this is the zero-element of $Cl(k[M])$. Since the mentioned element trivializes in $Cl(k[M]_z)$, it must already be trivial before localization. $\square$

For a finite convex polyhedron P and its vertex v, we say that a facet $F \subset P$ (i.e., a face of dimension $\dim(P)-1$) is visible from v if $v \notin F$. We let $\#v^0$ denote the number of facets of P that are visible from v. Clearly, $\#v^0 = 1$ if and only if P is a pyramid for which v is a vertex.

$F(P)$ will refer to the set of all facets of P.

Lemma 2.6. *Let k be a field and M be a finitely generated normal monoid with $U(M)$ trivial. Then*

(a) *the natural map*

$$Cl(k[M]) \to Cl(k[M]_\mu)$$

is an isomorphism, where $\mu \subset k[M]$ is the ideal of augmentation of $k[M]$,

(b) *$rank(Cl(k[M])) = \#F(\Phi(M)) - rank(M)$.*

Proof. For (a) see the proof of Corollary 2 in [6] (see also [1, p. 475] and [21, p. 26]). (b) follows from Theorems 16.7 and 16.9 in [11] (for the direct geometrical proof see [16, Part 2]). $\square$

Lemma 2.7. *Let k be a field and M and N be two finitely generated normal monoids with $U(M)$ and $U(N)$ trivial. Assume $f: k[M] \to k[N]$ is an isomorphism of k-algebras and μ is the augmentation ideal of $k[M]$. Then $N \backslash f(\mu)$ is a free submonoid of N, each basic element of which splits N.*

Proof. Let $v_1, \dots, v_n$ denote the vertices of $\Phi(N)$. By the finite generation and the normality of N, each of the submonoids

$$N(\{v_1\}), \dots, N(\{v_n\}) \subset N$$

is isomorphic to $\mathbb{Z}_+$. We denote the corresponding generators by $x_1, \dots, x_n$, respectively. $\square$

Claim 1. *If $x_i \notin f(\mu)$, then x_i splits N.*

Proof. Assume $x_i \notin f(\mu)$. First we show that $\#v_i^0 = 1$. Suppose $\#v_i^0 > 1$. We have the commutative diagram

$$(*) \qquad \begin{tikzcd} k[M] \arrow[r, "f"] \arrow[dd] & k[N] \arrow[dd] \arrow[dr] & \\ & & k[N]_{x_i} \arrow[dl] \\ k[M]_\mu \arrow[r] & k[N]_{f(\mu)} & \end{tikzcd}$$

So by Lemma 2.6(a) we get

$$Cl(k[N]) \to Cl(k[N]_{x_i})$$

is an isomorphism. On the other hand, by Lemma 2.3

$$k[N]_{x_i} \approx k[N'][X^{\pm 1}],$$

where X is a variable and N' is a finitely generated normal monoid with $U(N')$ trivial. Moreover, by the same lemma and by the assumption $\#v_i^0 > 1$, we get

$$\#F(\Phi(N')) \leq \#F(\Phi(N)) - 2.$$

We have

$$Cl(k[N]_{x_i}) = Cl(k[N'][X^{\pm 1}]) = Cl(k[N'])$$

(for the standard facts on Cl see [2, Chap. VII]), which by Lemma 2.6 (b) implies

$$\begin{aligned}\#F(\Phi(N)) - rank(N) &= rank(Cl(k[N])) = rank(Cl(k[N']))\\ &= \#F(\Phi(N')) - rank(N') \leq \#F(\Phi(N)) - rank(N) - 1\end{aligned}$$

(recall that $rank(N') = rank(N) - 1$). This contradiction shows $\#v_i^0 = 1$. Therefore, $\Phi(N)$ is a pyramid for which v_i is a vertex.

Now the same diagram $(*)$ and Lemma 2.5 completes the proof of the claim. □

It follows immediately from the above claim that we will be done when we show the next

Claim 2. *Let $x_{i_1}, \ldots, x_{i_k}$ be those x_i which do not belong to $f(\mu)$. Then $N \backslash f(\mu)$ is the submonoid of N generated by $\{x_{i_1}, \ldots, x_{i_k}\}$.*

Proof. First of all $f(\mu)$ is a maximal ideal of $k[N]$. It follows that the submonoid of N generated by $\{x_{i_1}, \ldots, x_{i_k}\}$ is contained in $N \backslash f(\mu)$. We denote this submomoid by $[x_{i_1}, \ldots, x_{i_k}]$.

Now observe that for any element $y \in N$ there exists $c \in \mathbb{N}$ such that y^c belongs to the submonoid of N generated by $\{x_1, \ldots, x_n\}$, which we denote by $[x_1, \ldots, x_n]$. It is clear from the previous claim that

$$y^c \in [x_{i_1}, \ldots, x_{i_k}] \;\Leftrightarrow\; y \in [x_{i_1}, \ldots, x_{i_k}].$$

Assume $y \notin [x_{i_1}, \ldots, x_{i_k}]$. Then by the remarks above there exists $c \in \mathbb{N}$ for which

$$y^c \in [x_1, \ldots, x_n] \backslash [x_{i_1}, \ldots, x_{i_k}].$$

Therefore, y^c is divisible by an element from $\{x_1,\ldots,x_n\}\backslash\{x_{i_1},\ldots,x_{i_k}\}$. Then $y^c \in f(\mu)$. Hence $y \in f(\mu)$. That means

$$N\backslash f(\mu) = [x_{i_1},\ldots,x_{i_k}]. \qquad \square$$

Lemma 2.8. *Let k be a field and M and N be two finitely generated normal monoids with $U(M)$ and $U(N)$ trivial. If $k[M]$ and $k[N]$ are isomorphic as k-algebras, then they are isomorphic as augmented k algebras.*

Proof. Let μ denote the augmentation ideal of $k[M]$ (that corresponds to the monoid M). Assume

$$f : k[M] \to k[N]$$

is a k-algebra isomorphism. By Lemma 2.7 there exists a submonoid $N_0 \subset N$ and elements $x_1,\ldots,x_k \in N\backslash\{1\}$ which are transcendental over $k[N_0]$ such that

$$k[N] = k[N_0][x_1,\ldots,x_k]$$

and

$$N_0\backslash\{1\} \subset f(\mu).$$

Let $a_1,\ldots,a_k \in k$ denote the constant terms of the elements $f^{-1}(x_1),\ldots,f^{-1}(x_k) \in k[M]$, respectively (i.e., a_i is the image of $f^{-1}(x_i)$ under the augmentation map $k[M] \to k$). Denote by g the $k[N_0]$-automorphism of $k[N]$ defined by

$$g(x_i) = x_i - a_i, \quad i \in [1,k].$$

Then the constant terms of the elements $f^{-1}g(x_1),\ldots,f^{-1}g(x_k) \in k[M]$ are all zero, in other words, $f^{-1}g : k[N] \to k[M]$ is an augmented isomorphism of k-algebras. $\square$

Step II: Let k be a field. An algebraic torus of dimension $r \in \mathbb{N}$ is an algebraic group isomorphic to $\mathbb{T}_r = U(k)^r$, where $U(k)$ is the multiplicative group of k. Assume A is an affine k-algebra. It will be identified with its natural image in $S^{-1}A$ for any multiplicative subset $S \subset A\backslash\{0\}$ (by an affine algebra we mean a finitely generated k-algebra which is a domain).

An *embedded torus* (in *Spec*(A), or for A) is a pair $\tau = (S, \bar{a})$ where $S \subset A\backslash\{0\}$ is a multiplicative subset and $\bar{a} = \{a_1,\ldots,a_r\} \subset S^{-1}A$ is an algebraically independent subset over k for which the following conditions are satisfied:

(a) $S^{-1}A = k[a_1^{\pm 1},\ldots,a_r^{\pm 1}]$,

(b) A is generated as a k-algebra by a certain finite system of Laurent monomials in a_i.

(Unless the contrary is stated explicitly, we always mean ‘pure’ monomials, i.e., those with scalar factor 1.)

Assume $X_1,\dots,X_r$ are variables. The condition (b) above means that A is an isomorphic image of some monomial subalgebra of the Laurent polynomial ring $k[X_1^{\pm 1},\dots, X_r^{\pm 1}]$ under the map

$$k[X_1^{\pm 1},\dots,X_r^{\pm 1}] \to S^{-1}A$$

determined by $X_i \mapsto a_i$.

Observe that we do *not* require normality of A. Observe also that $r = \dim(A)$.

Later on $M(\bar{a})$ will refer to the multiplicative submonoid of A which consists of all those elements that are (Laurent) monomials in a_i. Thus,

$$A = k[M(\bar{a})]$$

in the sense that $M(\bar{a})$ spans A as k-algebra. Simultaneously, A can be thought of as the monoid k-algebra corresponding to the monoid $M(\bar{a})$ (this justifies our notation).

Clearly, the two conditions $U(A) = U(k)$ and $U(M(\bar{a})) = 1$ coincide.

Given a k-algebra A, G_A will denote the group of all k-algebra automorphisms of A.

Let $\tau_1 = (S_1, \bar{a})$ and $\tau_2 = (S_2, \bar{b})$ be two embedded tori in *Spec*(A). Assume $g \in G_A$. We shall write $g^*\tau_1 = \tau_2$ if g can be extended to a k-algebra isomorphism between $S_2^{-1}A$ and $S_1^{-1}A$:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "\cap"'] & A \arrow[d, hook, "\cap"] \\ S_2^{-1}A \arrow[r, dashed] & S_1^{-1}A. \end{tikzcd}$$

Any embedded torus $\tau = (S, \bar{a})$ defines an embedding

$$\mathbb{T}_r \to G_A$$

in a natural way. Namely, we let $\mathbb{T}_r$ act on A as follows. Any element $\xi = (\xi_1,\dots,\xi_r) \in \mathbb{T}_r$ defines a k-automorphism of $S^{-1}A$ by putting

$$a_i \mapsto \xi_i a_i, \quad i \in [1,r].$$

This gives rise to the embedding

$$\mathbb{T}_r \to G_{S^{-1}A}$$

and each of these automorphisms restricts to a k-automorphism of A. Hence, the homomorphism of groups

$$\mathbb{T}_r \to G_A.$$

Finally, this homomorphism is injective because any automorphism of A can be lifted to one at most automorphism of $S^{-1}A$.

Later on we denote by τ^* the image of the corresponding embedding

$$\mathbb{T}_r \to G_A.$$

Lemma 2.9. *Let $\tau=(S,\bar{a})$ be an embedded torus for some affine k-algebra A (k a field). Then τ^* consists precisely of those k-automorphisms of A which act on elements of $M(\bar{a})$ by multiplication on scalars, i.e., $g\in\tau^*$ if and only if*

$$x\in M(\bar{a}) \Rightarrow g(x)=s_x x$$

for some $s_x\in U(k)$.

Proof. Trivial. □

Lemma 2.10. *Let k be an infinite field. Assume A is an affine k-algebra, $\tau_1=(S_1,\bar{a})$ and $\tau_2=(S_2,\bar{b})$ are two embedded tori for A and $g\in G_A$. Then the following three conditions are equivalent:*

(a) $g^*\tau_1=\tau_2$,

(b) $g^{-1}\tau_1^* g=\tau_2^*$ *(in G_A)*,

(c) $M(\bar{a})\approx M(\bar{b})$ *(as monoids)*.

Proof. (a)$\Rightarrow$(b). Assume $g^*\tau_1=\tau_2$. We shall use the same letter g for the extension of g to the localization $S_2^{-1}A$. Therefore, we have the commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[d, "\cap"'] & & A \arrow[d, "\cap"] \\
S_2^{-1}A \arrow[d, equal] & & S_1^{-1}A \arrow[d, equal] \\
k[b_1^{\pm1},\ldots,b_r^{\pm1}] & & k[a_1^{\pm1},\ldots,a_r^{\pm1}], \arrow[ll, "g^{-1}"']
\end{tikzcd}$$

where $\bar{b}=\{b_1,\ldots,b_r\}$ and $\bar{a}=\{a_1,\ldots,a_r\}$. Since any isomorphism between Laurent polynomial algebras must map monomials to monomials (in general not to 'pure' ones), just because monomials are the only units in these algebras, the diagram above and Lemma 2.9 immediately imply

$$g^{-1}\xi^* g\in\tau_2^*$$

for any $\xi^*\in\tau_1^*$. Hence $g^{-1}\tau_1^* g\subset\tau_2^*$. The inclusion $\tau_2^*\subset g^{-1}\tau_1^* g$ is equivalent to the inclusion $g\tau_2^* g^{-1}\subset\tau_1^*$, and the latter follows from the arguments above applied to the commutative diagram:

$$\begin{tikzcd} A \arrow[d, "\cap"'] & & A \arrow[ll, "g^{-1}"'] \arrow[d, "\cap"] \\ S_2^{-1}A \arrow[d, equal] & & S_1^{-1}A \arrow[d, equal] \\ k[b_1^{\pm 1},\dots,b_r^{\pm 1}] \arrow[rr, "g"] & & k[a_1^{\pm 1},\dots,a_r^{\pm 1}]. \end{tikzcd}$$

(b) $\Rightarrow$ (a). Assume $g^{-1}\tau_1^* g=\tau_2^*$. First let us show that g transforms monomials into monomials (of the general type), i.e., for any element $x\in M(\bar{b})$ its image $g(x)$ is of the type sy for some $s\in U(k)$ and $y\in M(\bar{a})$.

Assume to the contrary that there is an element $x\in M(\bar{b})$ such that in the canonical k-linear expansion of $g(x)$ with respect to elements of $M(\bar{a})$ there occur two distinct elements $y_1, y_2\in M(\bar{a})$. Since the field k is infinite one easily concludes that there exists $\xi^*\in\tau_1^*$ such that

$$\xi^*(y_1)=s_1y_1$$

and

$$\xi^*(y_2)=s_2y_2$$

for some *distinct* elements s_1 and s_2 of $U(k)$. Therefore, there does not exist $s\in U(k)$ for which

$$\xi^*(g(x))=sg(x), \quad x\in M(\bar{a}),$$

or equivalently, there does not exist $s\in U(k)$ for which

$$(g^{-1}\xi^* g)(x)=sx, \quad x\in M(\bar{a}).$$

But this contradicts the assumption $g^{-1}\tau_1^* g=\tau_2^*$ because of Lemma 2.9.

Having established the claim on *monomial-to-monomial* transformations, there only remains to notice that the unique extension of g to $S_2^{-1}A$ has its image in $S_1^{-1}A$, that is, $g^*\tau_1=\tau_2$.

(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). That $M(\bar{a})\approx M(\bar{b})$ is implied by (b) immediately follows from the aforementioned 'monomial-to-monomial' nature of g. Conversely, if $M(\bar{a})\approx M(\bar{b})$, then any monoid isomorphism between $M(\bar{a})$ and $M(\bar{b})$ extends to a k-automorphism of A that fits in the square required for (a). $\square$

Step III: Let k be a field, A an affine k-algebra and $\tau=(S,\bar{a})$ its embedded torus. A maximal ideal $\mu_\tau\subset A$ is called a *stable point* of τ if $U(M(\bar{a}))=1$ and μ_τ is generated by $M(\bar{a})\setminus\{1\}$. Thus μ_τ (if it exists, i.e., if $U(M(\bar{a}))=1$) is determined uniquely.

Clearly,

$$A/\mu_\tau = k.$$

The finite generation of the ideal μ_τ implies that of the k-linear space μ_τ/μ_τ^d for any natural number $d \geq 2$. We let

$$GL(\mu_\tau/\mu_\tau^d)$$

denote the group of automorphisms of this k-linear space. We also put

$$G_{A,\tau} = \{g \in G_A \mid g(\mu_\tau) = \mu_\tau\}.$$

$G_{A,\tau}$ is a subgroup of G_A and one has the natural group homomorphism

$$h_\tau^{(d)} : G_{A,\tau} \to GL(\mu_\tau/\mu_\tau^d)$$

for any natural number $d \geq 2$.

Put $K_\tau^{(d)} = Ker\,(h_\tau^{(d)})$.

Lemma 2.11. *For any embedded torus $\tau = (S, \bar{a})$ having a stable point and any natural number $d \geq 2$, the following hold:*

(a) $\tau^* \subset G_{A,\tau}$,

(b) $K_\tau^{(d)} \cap \tau^* = 1$.

Later on we will denote by $M_{<d}(\bar{a})$ the set of those elements of $M(\bar{a})$ which are presentable as products of at most $d-1$ non-unit elements from $M(\bar{a})$.

Proof. The inclusion is trivial. In order to show the equality above, observe that the space μ_τ/μ_τ^d can be thought of as the k-linear span of $M_{<d}(\bar{a})$. The latter in its order can be thought of as the k-linear subspace of A generated by $M_{<d}(\bar{a})$.

Now let $\xi_1^*, \xi_2^* \in \tau^*$ for some $\xi_1, \xi_2 \in \mathbb{T}_r$. If $h_\tau^{(d)}(\xi_1^*) = h_\tau^{(d)}(\xi_2^*)$, then ξ_1^* and ξ_2^* act identically on $M_{<d}(\bar{a})$. In particular, they act identically on the subset $M_{<2}(\bar{a}) \subset M_{<d}(\bar{a})$ which is nothing else but the minimal generating set (always uniquely determined) of $M(\bar{a})$. But, as remarked earlier – just before Lemma 2.9 – we then have $\xi_1^* = \xi_2^*$, that means $K_\tau^{(d)} \cap \tau^* = 1$. $\square$

Observe that $GL(\mu_\tau/\mu_\tau^d)$ and $h_\tau^{(d)}$ depend only on the stable point μ_τ and the natural number d. $G_{A,\tau}$ in its order depends on μ_τ only. Thus, if two embedded tori τ_1 and τ_2 have the same stable point, then they define the same homomorphism $h_{\tau_1}^{(d)} = h_{\tau_2}^{(d)}$ for any natural number d.

For simplicity of notation, we put

$$h^{(d)}(\tau) = h_\tau^{(d)}(\tau^*).$$

Lemma 2.12. *Let k be a field of infinite transcedence degree over its prime subfield. Assume A is an affine k-algebra and $\tau_1 = (S_1, \bar{a})$ and $\tau_2 = (S_2, \bar{b})$ are two embedded tori* *in Spec(A) which have the same stable point, say μ. Assume further $d \geq 2$ is a natural number. If $h^{(d)}(\tau_1) = h^{(d)}(\tau_2)$, then τ_1^* and τ_2^* are conjugate in G_A. Moreover, for all sufficiently large d, depending only on τ_1, the equality $h^{(d)}(\tau_1) = h^{(d)}(\tau_2)$ implies that τ_1^* and τ_2^* are conjugate in G_A by an element from $K_{\tau_1}^{(d)}$ $(= K_{\tau_2}^{(d)})$.*

Proof. As mentioned above the subsets $M_{<2}(\bar{a}) \subset M_{<d}(\bar{a})$ and $M_{<2}(\bar{b}) \subset M_{<d}(\bar{b})$ can be thought of as the minimal generating sets of the monoids $M(\bar{a})$ and $M(\bar{b})$ and as bases of the k-linear space μ/μ^2 simultaneously. In particular, we see that $M(\bar{a})$ and $M(\bar{b})$ have the save number of minimal (i.e., indecomposible) generators, say n.
Assume

$$\{x_1, \ldots, x_n\} = M_{<2}(\bar{a})$$

and

$$\{y_1, \ldots, y_n\} = M_{<2}(\bar{b}). \quad \square$$

Claim 1. *For each $i \in [1, n]$ there exist $j \in [1, n]$ and $s \in U(k)$ such that $y_i = sx_j \bmod \mu^d$.*

Proof. Both sets $M_{<d}(\bar{a})$ and $M_{<d}(\bar{b})$ constitute bases of the same k-linear space μ/μ^d. At the same time $M_{<2}(\bar{a})$ and $M_{<2}(\bar{b})$ constitute k-bases of μ/μ^2. Thus for each $i \in [1, n]$ there exist $j \in [1, n]$ and $s \in U(k)$ such that sx_j is a summand in the k-linear expansion of y_i modulo μ^2 in the basis $M_{<2}(\bar{a})$. But then sx_j is a summand in the k-linear expansion of y_i modulo μ^d in the basis $M_{<d}(\bar{a})$ as well. Our claim is exactly that there are no other summands. Assume to the contrary that there exists $x \in M_{<d}(\bar{a}) \setminus \{x_j\}$ which is involved in the aforementioned linear expansion. We have

$$x_j = \prod_{k=1}^{r} a_k^{p_k}$$

and

$$x = \prod_{k=1}^{r} a_k^{q_k}$$

for some distinct vectors

$$p = (p_1, \ldots, p_r)$$

and

$$q = (q_1, \ldots, q_r)$$

in $\mathbb{Z}^r$.

For any $\xi \in \mathbb{T}_r$ we let ξ^* denote the image of ξ in τ_1^* and ξ^{**} that in τ_2^*. For $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{T}_r$ we have

$$\xi^*(x_j) = \left(\prod_{k=1}^{r} \xi_k^{p_k}\right) x_j$$

and

$$\xi^*(x) = \left(\prod_{k=1}^{r} \xi_k^{q_k}\right) x$$

Since k is infinity, we easily see that there exists $\xi \in \mathbb{T}_r$ such that

$$\prod_{k=1}^{r} \xi_k^{p_k} \neq \prod_{k=1}^{r} \xi_k^{q_k}.$$

In this situation

$$\xi^*(y_i)/y_i \in Q(A) \setminus k,$$

where $Q(A)$ denotes the field of fractions of A.

On the other hand, by the conditions of the lemma there exists $\eta \in \mathbb{T}_r$ such that

$$\xi^*(y_i) = \eta^{**}(y_i).$$

But by Lemma 2.9 we see that

$$\eta^{**}(y_i)/y_i \in U(k),$$

a contradiction. Claim 1 has been proved. □

Applying Claim 1 to the other y_i, we see that there is an enumeration of the x_i's and y_j's for which

$$y_i = c_i x_i \bmod \mu^d$$

for some $c_i \in U(k)$. We fix such an enumeration.

Our condition $h^{(d)}(\tau_1) = h^{(d)}(\tau_2)$ implies the following:

(1) For any $\xi \in \mathbb{T}_r$ there exists $\eta \in \mathbb{T}_r$ such that

$$\xi^*(x_i)/x_i = \eta^{**}(y_i)/y_i, \quad i \in [1, n],$$

(the quotients are considered in $Q(A)$).

By Lemma 2.10, the first part of Lemma 2.12 is proved when it is shown that the elements x_i and y_i are subject to the same relations in the monoids $M(\bar{a})$ and $M(\bar{b})$, respectively.

Assume

$$x_i = \prod_{j=1}^{r} a_j^{p_{ij}}, \quad i \in [1, n],$$

and

$$y_i = \prod_{j=1}^{r} b_j^{q_{ij}}, \quad i \in [1,n].$$

We put

$$p_i = (p_{i1}, \dots, p_{ir}),$$

$$q_i = (q_{i1}, \dots, q_{ir}), \quad i \in [1,n].$$

Then $M(\bar{a})$ and $M(\bar{b})$ are isomorphic to the additive submonoids of $\mathbb{Z}^r$ generated by $\{p_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$, respectively. What we have to show is that there exists a $\mathbb{Q}$-automorphism of the rational vector space $\psi : \mathbb{Q}^r \to \mathbb{Q}^r$ such that $\psi(p_i) = q_i$ for $i \in [1,n]$.

Next we translate condition (1) above into terms of p_i and q_i as follows:

(2) For any $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{T}_r$ there exists $\eta = (\eta_1, \dots, \eta_r) \in \mathbb{T}_r$ such that $\xi^{p_i} = \eta^{q_i}$ for $i \in [1,n]$, where

$$\xi^{p_i} = \prod_{j=1}^{r} \xi_j^{p_{ij}}, \qquad \eta^{q_i} = \prod_{j=1}^{r} \eta_j^{q_{ij}}, \quad i \in [1,n].$$

Now the following claim completes the proof of the first part of our lemma.

Claim 2. *Let* $p_i = (p_{i1}, \dots, p_{in})$ *and* $q_i = (q_{i1}, \dots, q_{in})$ *be two arbitrary systems of non-zero elements of* $\mathbb{Z}^r$ $(i \in [1,n])$. *If they satisfy condition* (2) *above, then there is a* $\mathbb{Q}$*-linear automorphism* $\psi : \mathbb{Q}^r \to \mathbb{Q}^r$ *for which*

$$\psi(p_i) = \psi(q_i)$$

for $i \in [1,n]$.

Proof. Without loss of generality, we may assume

$$\{p_1, \dots, p_r\} \subset \mathbb{Q}^r$$

is a $\mathbb{Q}$-linear independent subset (observe that $r \le n$ with $r = n$ if and only if A is a polynomial ring over k). Since k has infinite transcendence degree over its prime subfield, we can pick an element

$$\xi = (\xi_1, \dots, \xi_r) \in \mathbb{T}_r$$

such that $\{\xi_1, \dots, \xi_r\}$ is an algebraically independent system over the mentioned prime subfield $k_0 \subset k$. In this situation the subset

$$\{\xi^{p_1}, \dots, \xi^{p_r}\} \subset U(k)$$

will be transcendental over k_0 as well. By condition (2)

$$\xi^{p_i} = \eta^{q_i}, \quad i \in [1,n],$$

for some $\eta \in \mathbb{T}_r$. In particular,

$$\{\eta^{q_1}, \dots, \eta^{q_r}\} \subset U(k)$$

is algebraically independent over k_0. The latter is equivalent to the following pair of conditions:

(a) $\{\eta_1, \dots, \eta_r\} \subset U(k)$ is algebraically independent subset over k_0, where $\eta = (\eta_1, \dots, \eta_r)$,

(b) $\{q_1, \dots, q_r\} \subset \mathbb{Q}^r$ is a $\mathbb{Q}$-linearly independent subset.

We need the condition (b) only. It implies that the bijective mapping

$$p_i \mapsto q_i, \quad i \in [1, r],$$

gives rise to a $\mathbb{Q}$-linear automorphism, say ψ, of $\mathbb{Q}^r$. Let us show that

$$\psi(p_i) = q_i$$

for $i \in [1, n]$.

For each $i \in [1, n]$ there exist $\lambda_{ij} \in \mathbb{Q}$, $j \in [1, r]$, such that

$$q_i = \sum_{j=1}^{r} \lambda_{ij} q_j.$$

Equivalently,

$$l_i q_i = \sum_{j=1}^{r} l_{ij} q_j, \quad i \in [1, n]$$

for some $l_{ij} \in \mathbb{Z}$ and $l_i \in \mathbb{Z} \setminus \{0\}$. It only remains to show that

$$l_i p_i = \sum_{j=1}^{r} l_{ij} p_j, \quad i \in [1, n].$$

But the latter equalities directly follow from the equalities

$$\xi^{p_i} = \eta^{q_i}, \quad i \in [1, n]$$

and the fact that $\xi_1, \dots, \xi_r$ are algebraically independent over k_0. □

We have completed the proof of the claim that τ_1^* and τ_2^* are conjugate in G_A. Now we show that the conjugating element can be chosen from $K_{\tau_1}^{(d)}$ $(= K_{\tau_2}^{(d)})$ if d is large enough with respect to τ_1.

The proof of Lemma 2.9 and the arguments above show that $x_i \mapsto y_i$ extends to a monoid isomorphism $g : M(\bar{a}) \xrightarrow{\approx} M(\bar{b})$ while $y_i = c_i x_i \bmod \mu^d$ for some $c_i \in U(k)$ $(i \in [1, n])$. We let the same letter g denote the corresponding isomorphism $k[M(\bar{a})] \xrightarrow{\approx} k[M(\bar{b})]$. It is clear that g induces the identity automorphism of μ/μ^d if and only if $c_i = 1$ for all i. That is, $g \in K_{\tau_1}^{(d)}$ if and only if $c_i = 1$ for all i.

We write

$$g : x_i \mapsto y_i = c_i x_i + m_i$$

for some $m_i \in \mu^d$. Since the monoid $M(\bar{a})$ is finitely generated, there is a finite system of 'basic' relations between the x_i, i.e., there is a finite system of pairs of monomials in new variables z_i, say

$$(W_j(z_1,\dots,z_n),\ V_j(z_1,\dots,z_n)), \quad j \in [1,J]$$

for some $J \in \mathbb{N}$, such that all the relations between the x_i are obtained (in the obvious sense) from the ones

$$W_j(x_1,\dots,x_n) = V_j(x_1,\dots,x_n), \quad j \in [1,J].$$

(The equations are considered in $M(\bar{a}) \subset A$; one just uses the fact that $k[M(\bar{a})]$ is Noetherian.)

For any element $\phi \in k[M(\bar{a})]$ we let *supp*(ϕ) denote the set of the elements from $M(\bar{a})$ that are involved in the canonical k-linear expansion of ϕ.

Claim 3. *For any finite system of monomials* $\{U_j(z_1,\dots,z_n)\}_j$ *there exists a natural number* d_0 *such that*

$$U_j(x_1,\dots,x_n) \in supp(U_j(s_1x_1 + l_1,\dots,s_nx_n + l_n))$$

for all indices j *and arbitrary elements* $s_i \in U(k)$ *whenever* $d > d_0$ *and* $l_i \in \mu^d$ $(i \in [1,n])$.

Proof. If d is large enough, then $\{U_j(x_1,\dots,x_n)\}_j \in M_{<d}(\bar{a})$. On the other hand, it is clear that $\mu^d \cap M(\bar{a}) \subset M(\bar{a}) \backslash M_{<d}(\bar{a})$. Therefore, for such natural numbers d the monomials $U_j(s_1x_1,\dots,s_nx_n)$ will survive in the canonical k-linear expansions of $U_j(s_1x_1 + l_1,\dots,s_nx_n + l_n)$, respectively. Hence the claim. □

Now we complete the proof of Lemma 2.12 as follows. We know that

$$W_j(y_1,\dots,y_n) = V_j(y_1,\dots,y_n), \quad j \in [1,J].$$

Using Claim 3 we come to the conclusion that

$$W_j(c_1,\dots,c_n) = V_j(c_1,\dots,c_n), \quad j \in [1,J],$$

providing d is large enough with respect to τ_1. In this situation $x_i \mapsto c_i x_i$ defines an element of τ_1^*, say ξ^*. Then (as the proof of Lemma 2.9 shows) $g(\xi^*)^{-1}$ is the desired element of $K_{\tau_1}^{(d)}$. □

Step IV: Let k be an algebraically closed field. One knows that if we are given two *distinct* algebraic tori (over k) $\mathbb{T} \subset \mathbb{T}'$ then $\#(\mathbb{T}'/\mathbb{T}) = \infty$. Indeed, since k is algebraically closed all tori are divisible. Then so is the quotient group $(\mathbb{T}'/\mathbb{T})$ and a non-zero divisible group is infinite.

Assume k is as above, d is a natural number, A is an affine k-algebra and τ its embedded torus having a stable point. By Lemma 2.11, $K_\tau^{(d)} \cap \tau^* = 1$. Thus, the restricition of $h_\tau^{(d)}$ to τ^* is an isomorphism between τ^* and $h^{(d)}(\tau)$. We put

$$H_\tau^{(d)} = Im(h_\tau^{(d)} : G_{A,\tau} \to GL(\mu_\tau/\mu_\tau^d))$$

(notation as above).

Lemma 2.13. *Let k, A and τ be as above. Then $h^{(d)}(\tau)$ is a maximal torus of $H_\tau^{(d)}$ for all sufficiently large $d \in \mathbb{N}$.*

Proof. Assume there is a torus $\mathbb{T} \subset H_\tau^{(d)}$ strictly containing $h^{(d)}(\tau)$. Then by the remark above

$$\#(\mathbb{T}/h^{(d)}(\tau)) = \infty.$$

Therefore,

$$\#(N_{H_\tau^{(d)}}(h^{(d)}(\tau))/h^{(d)}(\tau)) = \infty,$$

where for an extension of groups $A \subset B$ we let $N_B(A)$ denote the normalizer of A in B, i.e.,

$$N_B(A) = \{b \in B \mid b^{-1}Ab = A\}.$$

Clearly, if $f : C \to B$ is a surjective group homomorphism, then

$$f^{-1}(N_B(A)) = N_C(f^{-1}(A)).$$

Therefore, if $\#(N_B(A)/A) = \infty$, then $\#(N_C(f^{-1}(A))/f^{-1}(A)) = \infty$ providing f is surjective.

Applying the above formula to the homomorphism $h^{(d)}(\tau)$ and using the equality

$$(h_\tau^{(d)})^{-1}(h^{(d)}(\tau)) = K_\tau^{(d)}\tau^*,$$

which is a consequence of the normality of $K_\tau^{(d)}$ in $G_{A,\tau}$, we get

$$\#(N_{G_{A,\tau}}(K_\tau^{(d)}\tau^*)/K_\tau^{(d)}\tau^*) = \infty.$$

(Here we put $K_\tau^{(d)}\tau^* = \{g_1g_2 \mid g_1 \in K_\tau^{(d)},\ g_2 \in \tau^*\}$.)

Let $q_1, q_2, \ldots$ be elements of $N_{G_{A,\tau}}(K_\tau^{(d)}\tau^*)$ which represent different elements in

$$N_{G_{A,\tau}}(K_\tau^{(d)}\tau^*)/K_\tau^{(d)}\tau^*.$$

For each $i \in \mathbb{N}$ we have

$$K_\tau^{(d)}\tau^* = q_i^{-1}(K_\tau^{(d)}\tau^*)q_i = K_\tau^{(d)}(q_i^{-1}\tau^*q_i).$$

Put

$$t_i = q_i^{-1}\tau^*q_i.$$

Claim 1. *For each $i \in \mathbb{N}$ there exists an embedded torus τ_i (for A) which has the same stable point as τ and such that $\tau_i^* = t_i$.*

Proof. Assume $\tau = (S, \bar{a})$, $\bar{a} = \{a_1, \dots, a_r\}$. We put

$$S_i = q_i^{-1}(S)$$

and

$$\bar{a_i} = \{q_i^{-1}(a_1), \dots, q_i^{-1}(a_r)\}.$$

Then we have the following commutative square:

$$\begin{tikzcd} A \arrow[rr, "q_i"] \arrow[d, hook, "\cap"'] & & A \arrow[d, hook, "\cap"] \\ S_i^{-1}A \arrow[d, equal] & & S^{-1}A \arrow[d, equal] \\ k[(q_i^{-1}(a_1))^{\pm 1}, \dots, (q_i^{-1}(a_r))^{\pm 1}] \arrow[rr] & & k[a_1^{\pm 1}, \dots, a_r^{\pm 1}]. \end{tikzcd}$$

In other words, we have the embedded torus

$$\tau_i = (S_i, \bar{a}_i) \subset Spec(A)$$

and the equality

$$q_i^* \tau = \tau_i.$$

By Lemma 2.10 we get

$$t_i = q_i^{-1} \tau^* q_i = \tau_i^*.$$

It only remains to show that τ_i has the same stable point as τ. But the stable point of τ_i is the maximal ideal of A generated by $q_i^{-1}(M(\bar{a})) \backslash \{1\}$, that is,

$$\mu_{\tau_i} = q_i^{-1} \mu_\tau,$$

and we are done because $q_i \in G_{A,\tau}$. $\square$

Let τ_i be as above. Because of the equality

$$K_\tau^{(d)} \tau^* = K_\tau^{(d)} \tau_i^*,$$

we have $h^{(d)}(\tau^*) = h^{(d)}(\tau_i^*)$. So by Lemma 2.12, for each $i \in \mathbb{N}$ there exists $p_i \in K_\tau^{(d)}$ such that

$$p_i^{-1} \tau^* p_i = q_i^{-1} \tau^* q_i$$

providing $d \in \mathbb{N}$ is sufficiently large with respect to τ. That means $q_i p_i^{-1} \in N_{G_{A,\tau}}(\tau^*)$ for each $i \in \mathbb{N}$ (d as above).

Let us show that $q_i p_i^{-1}$ represent different elements in

$$N_{G_{A,\tau}}(\tau^*)/\tau^*.$$

Indeed, if $(q_i p_i^{-1})(q_j p_j^{-1})^{-1} \in \tau^*$ for some distinct i and j, then the inclusions $p_i, p_j \in K_\tau^{(d)}$ and the normality of $K_\tau^{(d)}$ in $G_{A,\tau}$ would imply

$$q_i q_j^{-1} \in K_\tau^{(d)} \tau^*,$$

a contradiction.

Thus having assumed $h^{(d)}(\tau)$ is not a maximal torus in $H_\tau^{(d)}$, we arrive at the conclusion

$$\#(N_{G_{A,\tau}}(\tau^*)/\tau^*) = \infty.$$

Now the following claim gives us the desired contradiction.

Claim 2. $\#(N_{G_{A,\tau}}(\tau^*)/\tau^*) < \infty$ *for any embedded torus* τ.

Proof. By Lemma 2.10 $N_{G_{A,\tau}}(\tau^*)$ consists precisely of those k-automorphisms g of A for which $g^*\tau = \tau$. One immediately sees that the equality $g^*\tau = \tau$ is equivalent to the requirement: $g(x)$ is of the type sy for any $x \in M(\bar{a})$, where $s \in U(k)$ and $y \in M(\bar{a})$. We define the map

$$\Theta : \mathbb{N}_{G_{A,\tau}}(\tau^*) \to Aut(M(\bar{a})),$$

where $Aut(M(\bar{a}))$ is the group of monoid automorphisms of $M(\bar{a})$, as follows:

$$(\Theta(g))(x) = y,$$

where $g(x) = sy$ for $x, y, \in M(\bar{a})$, $s \in U(k)$.

Straightforward arguments show that Θ is a group homomorphism and $Ker(\Theta) = \tau^*$. Therefore,

$$N_{G_{A,\tau}}(\tau^*)/\tau^* \approx Aut(M(\bar{a})).$$

Since $M(\bar{a})$ has a unique minimal generating set $Aut(M(\bar{a}))$ injects into the group of permutations of this minimal generating set. Hence $Aut(M(\bar{a}))$ is finite. $\square$

Step V: Now we are ready to prove Theorem 2.1.

Substep 1. (a) $\Rightarrow$ (b). If $R[M] \approx R[N]$ and M is normal then N is normal as well (just pass to some field of coefficients via scalar extension). So we have to show that if M and N are two finitely generated normal monoids, R is a ring and $R[M] \approx R[N]$ as R-algebras then $M \approx \mathbb{N}$.

Let us show that there exist two finitely generated normal monoids M_1 and N_1 only with trivial units such that $M \approx U(M) \times M_1$ and $N \approx U(N) \times N_1$.

Indeed, the normality condition implies $K(M)/U(M)$ is torsion free. Hence $U(M) \to K(M)$ is a split monomorphism (recall that $rank(M) < \infty$). Assume

$$K(M) = U(M) \oplus K_1.$$

Now it is straightforward to show that

$$U(M) \times (K_1 \cap M) \approx M$$

and that $K_1 \cap M$ is a finitely generated normal monoid with trivial units. The same arguments apply to N.

Assume we are given an R-isomorphism

$$f : R[M] \to R[N].$$

By a scalar-extension we can assume R is a field, say k. Due to the aforementioned remarks f can be rewritten as follows:

$$f : k[U(M)][M_1] \to k[U(N)][N_1]$$

for M_1 and N_1 as above. One easily shows that

$$U(k[U(M)][M_1]) = U(k) \oplus U(M)$$

and

$$U(k[U(N)][N_1]) = U(k) \oplus U(N).$$

(see [11, Theorem 1.11], for instance.)

Since f preserves units we conclude that f restricts to a k-algebra isomorphism between $k[U(M)]$ and $k[U(N)]$. Clearly, $rank(U(M)) = rank(U(N))$.

Let R_1 denote $k[U(M)]$. Identifying $k[U(N)]$ with R_1 (via the just mentioned isomorphism) we obtain the R_1-algebra isomorphism

$$g : R_1[M_1] \to R_1[N_1].$$

By a suitable scalar-extension we can pass to a k_1-algebra isomorphism

$$h : k_1[M_1] \to k_1[N_1].$$

for some field k_1. By Lemma 2.8 we can assume h is an augmented k_1-algebra isomorphism. Then (a) implies $M_1 \approx N_1$. Since we also have $U(M) \approx U(N)$ (they both are free abelian groups of the same rank), we obtain the desired isomorphism $M \approx N$.

Substep 2: (a) $\Rightarrow$ (c). First we have to define homogeneous monoids.

Definition 2.14. A finitely generated monoid M is called homogeneous if M is isomorphic to an additive submonoid of $\mathbb{R}^{r+1}$ (for some nonnegative integer r) which is generated by a finite system of elements of the type $(x, 1) \in \mathbb{R}^{r+1}$ where $x \in \mathbb{Z}^r$.

Clearly, homogeneous monoids have no nontrivial units and the only rank 1 homogeneous monoids are those isomorphic to $\mathbb{Z}_+$. The name 'homogeneous' comes from the observation that for such a monoid M and a ring R the monoid ring $R[M]$ carries naturally a graded structure where all the minimal generators of M have degree 1.

One easily observes that a monoid M is homogeneous if and only if it is finitely generated and there exists a codimension 1 hyperplane H in the real space $\mathbb{R}\otimes K(M)$ that avoids the origin and contains a generating set of M as a set of rational points in some Euclidean coordinate system of H. Moreover, the mentioned generating set is automatically the minimal generating set of M.

In the special case, the mentioned minimal generating set of M consists of all lattice points in some finite convex lattice polyhedron $P\subset H$ with respect to some Euclidean coordinate system of H we obtain exactly what was called a *polytopal semigroup* in [4].

Here is one more alternative definition of a homogeneous monoid: a monoid M is homogeneous iff it is finitely generated, has trivial $U(M)$ and the convex hull in $\mathbb{R}\otimes K(M)$ of the minimal generating set of M is a (finite, convex) polyhedron of dimension $rank(M)-1$.

Let M be a homogeneous monoid, $\{x_1,\dots,x_n\}$ be its minimal generating set and P the aforementioned convex hull. An element $x_i\in\{x_1,\dots,x_n\}$ will be called *extremal* if it is a vertex of P.

Lemma 2.15. *Let k be a field and M and N two finitely generated monoids. Assume $k[M]\approx k[N]$ and M is homogeneous. Then $U(N)=1$ and $k[M]\approx k[N]$ as augmented k-algebras.*

Proof. Assume

$$f: k[N]\to k[M]$$

is a k-algebra isomorphism. This isomorphism can be extended to the normalizations of the domains $k[M]$ and $k[N]$. But it is an immediate consequence of the general property of normalizations, we mentioned in the beginning of the section, that

$$\overline{k[M]}=k[\bar{M}]$$

and

$$\overline{k[N]}=k[\bar{N}],$$

where the bar refers to the appropriate normalization. Now by Lemma 2.2 both $\bar{M}$ and $\bar{N}$ are finitely generated normal monoids (clearly, with only trivial units). For a monoid L, which is finitely generated and has not non-trivial units, we let $gen(L)$ denote its minimal generating set. Observe that homogeneuity of M implies

$$gen(M)\subset gen(\bar{M}).$$

By Lemma 2.7, $\bar{M}\setminus\bar{f}(\bar{v})$ is a free submonoid of $\bar{M}$ each basic element of which splits $\bar{M}$, where $\bar{f}$ is the aforementioned extension of f and $\bar{v}$ is the augmentation ideal of

$k[\bar{N}]$ (with respect to the monoid $\bar{N}$). Obviously,

$$gen(\bar{M} \setminus \bar{f}(\bar{v})) \subset gen(\bar{M}).$$

Easy arguments ensure that any element of $gen(\bar{M} \setminus \bar{f}(\bar{v}))$ is a generator of a submonoid of $\bar{M}$ corresponding to some edge of $C(\bar{M})$. But $C(M) = C(\bar{M})$. These remarks imply that $gen(\bar{M} \setminus \bar{f}(\bar{v}))$ is a subset of all extremal elements in $gen(M)$. Therefore, $M \setminus f(v)$ is a free submonoid of M each basic element of which splits M. Now the same arguments as in the proof of Lemma 2.8 show that there is an augmented k-algebra homomorphism $g : k[M] \to k[N]$. $\square$

After Lemma 2.15 the implication (a) $\Rightarrow$ (b) becomes obvious.

Substep 3: Here we prove the claim (a).

Let R, M and N be as in Theorem 2.1 (a). By a suitable scalar extension we can pass to a k-algebra isomorphism

$$f : k[M] \to k[N]$$

for some algebraically closed field of infinite transcedence degree over its prime subfield.

Put $A = k[N]$. Let $\bar{a} = \{a_1, \ldots, a_r\}$ and $\bar{b} = \{b_1, \ldots, b_r\}$ be two arbitrary basis' of the free abelian groups $K(N)$ and $K(f(M))$ respectively (considered as multiplicative subgroups of the field of fractions of A).

In this situation we are given the two embedded tori in $Spec(A)$

$$\tau_1 = (N, \bar{a})$$

and

$$\tau_2 = (f(M), \bar{b}).$$

That f is augmented precisely means τ_1 and τ_2 have the same stable point, say μ. Let $d \geq 2$ be a natural number and let $H_0^{(d)}$ denote the subgroup of $H_{\tau_1}^{(d)}$ $(= H_{\tau_2}^{(d)})$ generated by $h^{(d)}(\tau_1)$ and $h^{(d)}(\tau_2)$ (notation as in Step IV). As remarked just before Lemma 2.13 $h^{(d)}(\tau_1)$ and $h^{(d)}(\tau_2)$ are two r-dimensional tori in $GL(\mu/\mu^d)$.

Claim. *$h^{(d)}(\tau_1)$ and $h^{(d)}(\tau_2)$ are algberaic tori in $GL(\mu/\mu^d)$.*

Proof. Let τ be any embedded torus in $Spec(A)$ with a stable point v. We will show that $h^{(d)}(\tau)$ is an algebraic torus in $GL(v/v^d)$ (for any natural number $d \geq 2$). We know that v/v^d can be thought of as the k-linear subspace of A spanned by $M_{<d}(\bar{c})$, where we put $\tau = (S, \bar{c})$. Using Lemma 2.9 we see that in this basis $h^{(d)}(\tau)$ consists of diagonal automorphisms of $GL(v/v^d)$. Fixing the mentioned basis we fix the corresponding identification

$$GL(v/v^d) = GL_m(k)$$

for $m = \#S_{<d}(\bar{c})$. In this way the group $h^{(d)}(\tau)$ is identified with a subgroup of $diag_m(k) = \mathbb{T}_m$.

The homomorphism $h_\tau^{(d)} : \tau \to \mathbb{T}_m$ can be written up explicitly. Assume

$$M_{<d}(\bar{c}) = \{x_1, \dots, x_m\}$$

for some monomials

$$x_i = c_1^{p_{i1}} \cdots c_r^{p_{ir}}, \quad i \in [1, m],$$

where we assume $\bar{c} = \{c_1, \dots, c_r\}$. Then $h_\tau^{(d)}$ is given by

$$(\xi_1, \dots, \xi_r) \mapsto (\xi_1^{p_{11}} \cdots \xi_r^{p_{1r}}, \dots, \xi_1^{p_{m1}} \cdots \xi_r^{p_{mr}}).$$

We, in particular, see that $h_\tau^{(d)}$ is an algebraic map. We know also that $h_\tau^{(d)}$ is injective. Hence the claim. □

Now to complete the proof of Theorem 2.1 (a) we need the following fact.

Proposition 2.16. *Any pair of connected algebraic subgroups of an algebraic group generates a closed (and connected) subgroup.*

Proof. This proposition is a special case of Proposition 7.5 in [19]. □

Returning to our situation we conclude from Proposition 2.16 that $H_0^{(d)}$ is Zariski closed in $GL(\mu/\mu^d)$. By Lemma 2.13 the two tori $h^{(d)}(\tau_1)$ and $h^{(d)}(\tau_2)$ are maximal in $H_{\tau_1}^{(d)}$ $(= H_{\tau_2}^{(d)})$ whenever d is sufficiently large. Therefore, they are maximal in $H_0^{(d)}$. But by *Borel's Theorem* all maximal tori in a linear algebraic group are conjugate [19, Corollary 21.3(A)]. So for d large there exists $\gamma \in H_0^{(d)}$ such that

$$\gamma^{-1} h^{(d)}(\tau_1) \gamma = h^{(d)}(\tau_2).$$

Assume $\gamma = h_{\tau_1}^{(d)}(g)$ for some $g \in G_{A,\tau_1}$ and consider the subgroup

$$g^{-1} \tau_1^* g \subset G_{A,\tau_1}.$$

As in Claim 1 in the proof of Lemma 2.13, there exists an embedded torus $\tau_0 = (S, \bar{c})$ in $Spec(A)$ for which

$$\tau_0^* = g^{-1} \tau_1^* g$$

and which has the same stable point as τ_1. We have $h^{(d)}(\tau_0) = h^{(d)}(\tau_2)$. On the other hand, by Lemma 2.10 (c) the two monoids $M(\bar{c})$ and $M(\bar{a})$ are isomorphic. Simultaneously, by Lemma 2.12 there exists $g_1 \in G_{A,\tau_1}$ such that

$$g^{-1} \tau_0^* g = \tau_2^*.$$

Then again by Lemma 2.10 (c) the two monoids $M(\bar{c})$ and $M(\bar{b})$ are isomorphic. Thus, finally, we have

$$M \approx f(M) = M(\bar{b}) \approx M(\bar{c}) \approx M(\bar{a}) = N.$$

Comments 2.17. (a) Lemma 2.8 is equivalent to the claim in Step 3 of [9]. However the proof as presented in [9] is 'spurious' (Math. Rev. 84f: 36). Our arguments via divisor class groups are completely different.

(b) As mentioned in the introduction we use $GL(\mu_\tau/\mu_\tau^d)$ instead of Demushkin's use of $GL(\mu_\tau/\mu_\tau^2)$. The point is that the arguments in the proof of Lemma 2.12 applied to $GL(\mu_\tau/\mu_\tau^2)$ would only show that τ_1^* and τ_2^* are conjugate in G_A by an element g which implies the k-linear transformation of $\mu_{\tau_1}/\mu_{\tau_1}^2$ that corresponds in the basis $\{x_1,\ldots,x_n\}$ to the matrix $diag(c_1,\ldots,c_n)$ (notation as in the proof of the mentioned lemma). But it is *by no means* clear that $x_i \mapsto c_i x_i$ defines an element of τ_1^* (as one actually has for d sufficiently large). We do not know whether it can be shown independently that the conjugating element g can actually be chosen from $K_{\tau_1}^{(2)}$ $(=K_{\tau_2}^{(2)})$. Recall that we need the inclusion $g \in K_{\tau_1}^{(2)}$ in order to show that $h^{(d)}(\tau)$ is a maximal torus of $H_\tau^{(d)}$.

(c) We have changed Demushkin's arguments in the first part of the proof of Lemma 2.12 as well (the previous ones seemed non-convincing).

(d) Step IV here is essentially equivalent to Steps 4 and 8 in [9] modulo corrections of several 'inessential improperties' (Math. Rev. 84f:36).

(e) Demushkin claimed in [9, Step 5] that for any embeded torus τ having a stable point μ_τ the subgroup $H_\tau^{(2)} \subset GL(\mu_\tau/\mu_\tau^2)$ is a (Zariski) closed subgroup. It is just mentioned in [9] that this directly follows from looking at the relations between generators of the monoid $M(\bar{a})$ (we assume $\tau = (S, \bar{a})$). But it is our opinion that this claim is not clear at all! (This is not clear equally for any natural d.) Moreover, it is precisely the description of $H_\tau^{(2)}$ in the terms close in spirit to 'looking at the relations in a monoid' for the *special case of rank 2 monoids* that constitutes the most difficult and lengthy sections in [18]. (As a result there is obtained certain information in [18] on the group of k-algebra automorphisms of $k[M]$ for the 2-dimensional case and this information suffices to settle the special case of the isomorphism problem for rank 2 monoids.) It should also be mentioned that the present proof of the general case avoids any reference to the structure of the automorphism group of $k[M]$. The point here is that this approach needs closedness in $GL(\mu_\tau/\mu_\tau^d)$ of only the subgroup of $H_\tau^{(d)}$ generated by certain pair of algebraic tori, and here general arguments (Proposition 2.16) work. I am grateful to B. Totaro for drawing my attention to this observation.

(f) From the point of view of the comments above it is natural to ask the following two questions:

Question 1: Assume we are given a finitely generated monoid M with $U(M)$ trivial and a field k. Let $\{x_1,\ldots,x_n\}$ be the minimal generating set of M and $g: k[M] \to k[M]$ be a k-algebra automorphism such that $g(x_i) = c_i x_i + m_i$ for some $c_i \in k \setminus \{0\}$ and

$m_i \in \mu^2$ $(i \in [1,n])$, where μ is the maximal ideal of $k[M]$ generated by $M \setminus \{1\}$. Is it true that $x_1^{u_1} \dots x_n^{u_n} = x_1^{v_1} \dots x_n^{v_n}$ implies $c_1^{u_1} \dots c_n^{u_n} = c_1^{v_1} \dots c_n^{v_n}$ $(u_1, \dots, u_n, v_1 \dots, v_n \in \mathbb{Z}_+)$?

Question 2: Is the subgroup $H_\tau^{(2)} \subset GL(\mu_\tau/\mu_\tau^2)$ a (Zariski) closed subroup? (Notation as in (e).)

The reader can easily observe that the both questions have positive answers in the special case of homogeneous monoids.

3. Application to discrete Hodge algebras

Let R be a (commutative) ring and $X_1, \dots, X_m$ be variables. An ideal $I \subset R[X_1, \dots, X_m]$, which is generated by a system of monomials (i. e., of those of the type $X_1^{a_1} \dots X_m^{a_m}$, $a_i \in \mathbb{Z}_+$), is called a *monomial* ideal. A *discrete Hodge algebra* is defined as an algebra of the type $R[X_1, \dots, X_m]/I$, where I is a monomial ideal (De Concini, et al. [8]).

Theorem 3.1. *Let R be a ring and $\{X_1, \dots, X_m\}$, $\{Y_1, \dots, Y_n\}$ be two collections of variables. Assume $I \subset R[X_1, \dots, X_m]$ and $J \subset R[Y_1, \dots, Y_n]$ are monomial ideals such that $\{X_1, \dots, X_m\} \cap I = \emptyset$, $\{Y_1, \dots, Y_n\} \cap J = \emptyset$ and*

$$R[X_1, \dots, X_m]/I \approx R[Y_1, \dots, Y_n]/J$$

as R-algebras. Then $m = n$ and there exists a bijective mapping

$$\Theta : \{X_1, \dots, X_m\} \to \{Y_1, \dots, Y_m\}$$

transforming I into J.

Theorem 3.1 has the following equivalent formulation.

Theorem 3.2. *Let R, X_i, Y_j, I and J be as above and*

$$R[X_1, \dots, X_m]/I \approx R[Y_1, \dots, Y_n]/J$$

as R-algebras. Then $m = n$ and there exists a bijective mapping

$$[\Theta] : \{[X_1], \dots, [X_m]\} \to \{[Y_1], \dots, [Y_m]\}$$

which gives rise to an R-algebra isomorphism between the two discrete Hodge algebras $R[X_1, \dots, X_m]/I$ and $R[Y_1, \dots, Y_n]/J$, where $[X_i]$ and $[Y_j]$ refer to the corresponding residue classes.

In the proof we shall need the following:

Lemma 3.3. *Let A be a commutative ring and X be a variable. Assume $\beta \subset A[X]$, $\beta \neq A[X]$, is an ideal generated by a system of 'non-pure' monomials, i.e. β is* *generated by elements of the type aX^d for some $a \in A$ and $d \in \mathbb{Z}_+$. If the residue class $[X]$ of X in $A[X]/\beta$ is a non-zero-divisor, then β is generated as an ideal by $\alpha = A \cap \beta$.*

Proof. We have to show that $\beta \subset \alpha A[X]$. That means we have to show $aX^d \in \alpha A[X]$ whenever $aX^d \in \beta$. But if $aX^d \in \beta$ then $[a][X]^d = 0$ and by the condition that $[X]$ is not a zero-divisor we get $[a] = 0$, i.e. $a \in \alpha$ ($[a]$ refers to the corresponding residue class). $\square$

Proof of Theorem 3.1. Using the 'scalar extension trick', without loss of generality, we can assume R is a field (integral domain for the arguments below would actually suffice).

Assume

$$f : R[X_1, \dots, X_m]/I \to R[Y_1, \dots, Y_n]/J$$

is an R-algebra isomorphism. Any element $\phi \in R[Y_1, \dots, Y_n]$ admits a unique canonical presentation as an R-linear form of 'pure' monomials in the Y_j outside J. We let $\phi(0)$ denote the constant term (the term of degree 0) of this expansion. Now, we put

$$\{i_1, \dots, i_k\} = \{1 \le i \le m | (f([X_i]))(0) \ne 0\}.$$

In this situation the elements $f([X_{i_1}]), \dots, f([X_{i_k}])$ are not zero-divisors of $R[Y_1, \dots, Y_n]/J$. Since f is an isomorphism the elements $[X_{i_1}], \dots, [X_{i_k}]$ themselves are not zero-divisors in $R[X_1, \dots, X_m]/I$. By (an iterated use of) Lemma 3.3 we then easily conclude that the ideal I is generated by the intersection

$$I_0 = I \cap R[\{X_i\}_{i \notin \{i_1, \dots, i_k\}}].$$

Therefore there exists a system of pure monomials in X_i, $i \notin \{i_1, \dots, i_k\}$ that generates I. Hence, we can write

$$R[X_1, \dots, X_m]/I = A/I_0[X_{i_1}, \dots, X_{i_k}],$$

where $A = R[\{X_i\}_{i \notin \{i_1, \dots, i_k\}}]$. So we can consider the A/I_0-algebra automorphism

$$g : R[X_1, \dots, X_m]/I \to R[X_1, \dots, X_m]/I$$

induced by

$$X_{i_1} \mapsto X_{i_1} - (f(X_{i_1}))(0),$$

$$\dots$$

$$X_{i_k} \mapsto X_{i_k} - (f(X_{i_k}))(0).$$

Then the R-algebra isomorphism

$$fg : R[X_1, \dots, X_m]/I \to R[Y_1, \dots, Y_n]/J$$

will be an isomorphism of augmented R-algebras, where we endow the two discrete Hodge algebras the augmented R-algebra structures induced by

$$[X_i] \mapsto 0, \quad i \in [1,m]$$

and

$$[Y_j] \mapsto 0, \quad j \in [1,n],$$

respectively. Denote the corresponding augmentation ideals by μ and ν, respectively. Since $R[X_1,\dots,X_m]/I$ carries the graded R-algebra structure determined by

$$deg([X_i]) = 1, \quad i \in [1,m]$$

(and similarly for $R[Y_1,\dots,Y_n]/J$), we arrive at the (bottom) graded R-algebra isomorphism

$$\begin{tikzcd}
 & gr_\mu(R[X_1,\dots,X_m]/I) \arrow[r, "gr(fg)"] & gr_\nu(R[Y_1,\dots,Y_n]/J) & \\
R[X_1,\dots,X_m]/I \arrow[ur, "="] \arrow[rrr, dashed] & & & R[Y_1,\dots,Y_n]/J. \arrow[ul, "="']
\end{tikzcd}$$

Since

$$\{X_1,\dots,X_m\} \cap I = \emptyset = \{Y_1,\dots,Y_n\} \cap J,$$

there exists a unique graded R-algebra isomorphism h fitting in the commutative square with canonical horizontal epimorphisms.

$$\begin{tikzcd}
R[X_1,\dots,X_m] \arrow[r] \arrow[d, "h"'] & R[X_1,\dots,X_m]/I \arrow[d, "gr(fg)"] \\
R[Y_1,\dots,Y_n] \arrow[r] & R[Y_1,\dots,Y_n]/J.
\end{tikzcd}$$

This square, in-particular, immediately implies $m = n$.

Now for each natural number d we let I_d and J_d denote the ideals generated by

$$I \cup \{X_1^{\delta_1} \dots X_m^{\delta_m} | \delta_i \geq 0, \delta_1 + \dots + \delta_m = d\}$$

and

$$J \cup \{Y_1^{\delta_1} \dots Y_m^{\delta_m} | \delta_j \geq 0, \delta_1 + \dots + \delta_m = d\},$$

respectively.

Clearly, if

$$R[X_1,\dots,X_m]/I = R \oplus A_1 \oplus A_2 \oplus \cdots$$

and

$$R[Y_1,\dots,Y_m]/I = R\oplus B_1\oplus B_2\oplus\cdots$$

are the corresponding graded structures then

$$R[X_1,\dots,X_m]/I_d = R\oplus A_1\oplus\cdots\oplus A_{d-1}\oplus 0\oplus 0\oplus\cdots$$

and

$$R[Y_1,\dots,Y_m]/J_d = R\oplus B_1\oplus\cdots\oplus B_{d-1}\oplus 0\oplus 0\oplus\cdots$$

(with right-hand sides considered as graded algebras in the obvious way). From these observations it follows that for each $d\in\mathbb{N}$ we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_d \arrow[r] \arrow[d] & R[X_1,\dots,X_m] \arrow[r] \arrow[d] & R[X_1,\dots,X_m]/I_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_d \arrow[r] & R[Y_1,\dots,Y_m] \arrow[r] & R[Y_1,\dots,Y_m]/J_d \arrow[r] & 0
\end{tikzcd}$$

with exact rows and vertical R-algebra isomorphisms, where I_d and J_d are considered as R-algebras without units and the last two vertical maps are homomorphisms of graded R-algebras ($deg(X_i)=1=deg(Y_j)$ and $deg([X_i])=1=deg([Y_j])$).

We let M_d denote the multiplicative monoid of all those pure monomials in the X_i which belong to I_d. N_d is defined similarly with respect to J_d. The monoid algebras $R[M_d]$ and $R[N_d]$ will be identified with the corresponding monomial subalgebras of $R[X_1,\dots,X_m]$ and $R[Y_1,\dots,Y_m]$, respectively. Since $R[M_d]$ is a minimal (universal) unitary R-algebra containing I_d and $R[N_d]$ is that containing J_d, we have $R[M_d]\approx R[N_d]$ as *augmented* R-algebras.

Claim. *M_d and N_d are finitely generated monoids for all natural number d.*

Proof. $K(M_d)$ is the free abelian group of all Laurent monomials in the X_i. Since $X_1^d,\dots,X_m^d\in M_d$ the cone $C(M_d)$, spanned by M_d in $\mathbb{R}^m$, is the standard positive rectangular cone (we identify $\mathbb{R}\otimes K(M_d)$ with $\mathbb{R}^d$). Hence by Gordan's lemma (Lemma 2.2) M_d is finitely generated.

The same arguments apply to N_d. $\square$

Remark 3.4. Observe that the multiplicative monoid of all pure monomials in I needs not be finitely generated, that is the R-subalgebra of $R[X_1,\dots,X_m]$ generated by R and I is not in general finitely generated, despite the fact that I is always finitely generated as an ideal.

We continue the proof of Theorem 3.1.

The claim above and Theorem 2.1 (a) imply $M_d \approx N_d$ as monoids for all $d \in \mathbb{N}$. Since

$$X_1, \ldots, X_m \in K(M_d)$$

and

$$Y_1, \ldots, Y_m \in K(N_d),$$

we easily conclude that for each natural number d there exists a bijective mapping

$$\Theta_d : \{X_1, \ldots, X_m\} \to \{Y_1, \ldots, Y_m\},$$

which transforms M_d into N_d.

There exists an infinite strictly increasing sequence

$$d_1 < d_2 < \cdots$$

such that

$$\Theta_{d_1} = \Theta_{d_2} = \cdots .$$

Put $\Theta = \Theta_{d_1}$. Then Θ transforms M_{d_k} into N_{d_k} for all $k \in \mathbb{N}$. Therefore, Θ transforms $\bigcap_{k=1}^{\infty} M_{d_k}$ into $\bigcap_{k=1}^{\infty} N_{d_k}$. It only remains to observe that

$$\bigcap_{k=1}^{\infty} M_{d_k} = M$$

and

$$\bigcap_{k=1}^{\infty} N_{d_k} = N,$$

where M and N are the multiplicative monoids of all pure monomials inside I and J, respectively. This clearly finishes the proof. □

Example 3.5. Let V and W be two finite sets and Δ_V and Δ_W be two abstract simplicial complexes on the vertex sets V and W, respectively. Assume k is a field and consider the two Stanley–Reisner rings $k[\Delta_V]$ and $k[\Delta_W]$ (see [5, Ch. 5] for the definitions). If $k[\Delta_V] \approx k[\Delta_W]$ as k-algebras, then there exists a bijective mapping $V \to W$ transforming Δ_V into Δ_W. This directly follows from Theorem 3.2, and this special case of the 'isomorphism problem for discrete Hodge algebras' is considered in [3].

Example 3.6. Assume V, W, Δ_V, Δ_W and k are as in the previous example. Put

$$k\{\Delta_V\} = k[\Delta_V]/(\{v^2 | v \in V\})$$

and

$$k\{\Delta_W\} = k[\Delta_W]/(\{w^2 | w \in W\}),$$

where we identify the vertices with the corresponding elements in the Stanley–Reisner rings.

Jürgen Herzog asked whether

$$k\{\Delta_V\} \approx k\{\Delta_W\}$$

as k-algebras implies

$$\Delta_V \approx \Delta_W$$

(the latter isomorphism meant in the sense of the previous example). Theorem 3.1 gives the positive answer to this question as well.

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